Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON DUHAMEL'S THEOREM.

BY R. L. MOORE.

In these *Annals*, July, 1903, pages 161–178, Osgood discusses the following proposition which he refers to Duhamel:

Proposition I. "Let $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ be a sum of positive infinitesimals which approaches a limit when $n = \infty$. Let $\beta_1 + \beta_2 + \cdots + \beta_n$ be a second sum of infinitesimals which differ from the infinitesimals of the first sum by infinitesimals of higher order; *i. e.*, let

$$\operatorname{Lim} \frac{\beta_i}{\alpha_i} = 1.$$

Then the second sum approaches a limit when $n = \infty$, and this limit is the same as that of the first sum:

$$\operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n) = \operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n)."$$

Osgood calls attention to two objections to Proposition I, as it stands. First, whether it is true or not depends on what interpretation is given to the requirement that Lim $\beta_i/\alpha_i = 1$. Secondly, there are certain simple problems where the hypothesis of I is not completely satisfied though the conclusion holds. After discussing these points he formulates a substitute theorem* which is at once rigorously true and well adapted to application in the problems above mentioned.

In the present paper I propose a second form,† which is suitable for application to a still wider range of problems and at the same time seems to be even easier‡ to apply.

In many, if not all, applications of this theorem the numbers α_{in}§ and β_{in}§ are related to subdivisions of an interval or (more generally) to sub-sets of a limited point-set in m-dimensional space. Let E be such a point-set. For each value of the positive integer n let $E_{1n}, E_{2n}, \cdots, E_{nn}$, be non-overlapping sub-sets of E of interior measures $e_{1n}, e_{2n}, \cdots, e_{nn}$, respectively. Let us now consider the numbers α_{in} and β_{in} as being related (through the intermediation of their subscripts)[*] to E_{in} for each pair of values of n and i ($i = 1, \cdots, n$).

---

* Loc. cit., page 173.

† Theorem 2, page 162.

‡ This statement is subject to certain reservations which are indicated in a footnote on page 165.

§ I prefer to write $\alpha_{1n}, \alpha_{2n} \cdots \alpha_{nn}$ instead of $\alpha_1, \alpha_2 \cdots \alpha_n$. Double subscripts are evidently desirable.

Osgood suggests[†] that, in certain problems at least, a natural interpretation of the requirement that

$$\operatorname*{Lim}_{n=\infty} \frac{\beta_{in}}{\alpha_{in}} = 1$$

might be[‡] that, for every point P of the point-set E,

$$\operatorname*{Lim}_{n=\infty} \frac{\beta_{i_{P_n}n}}{\alpha_{i_{P_n}n}} = 1,$$

where i_{P_n} *signifies such a value of* i *that* $E_{i_{P_n}n}$ *shall contain the point* P.

But he gives[§] an independence example to show that, under this interpretation, I is not a true proposition. If one examines this independence example, however, it is to be observed that here there is no upper bound to the set of values of the ratio

$$\left|\frac{\beta_{in}}{\alpha_{in}}\right|.$$

I have found that if to the hypothesis of I there be added the requirement that such a *bound shall exist* and

$$\operatorname*{Lim}_{n=\infty} \frac{\beta_i}{\alpha_i} = 1$$

be given the *interpretation suggested above, I is thereby transformed into a true proposition.* Moreover this proposition remains true even if a set of points (P) of measure 0 be relieved from the requirement that

$$\operatorname*{Lim}_{n=\infty} \frac{\beta_{i_{P_n}n}}{\alpha_{i_{P_n}n}} = 1.$$

The following theorem holds true.

Theorem 2. HYPOTHESIS:[||] (*a*) *E is a limited point-set in a space of m-dimensions. $E_{1n}, E_{2n}, \cdots, E_{nn}$, are (for each value of the positive integer n) non-overlapping sub-sets of E of interior measures $e_{1n}, e_{2n}, \cdots, e_{nn}$, respectively. r_{in}, r_{in}' ($i = 1, \cdots, n$), are numbers such that the set $\{ |r_{in}' - r_{in}| \}$ is a bounded set, i. e., there exists a number c such that for all values of n and i*

---

[*] Thus, for example, α_{12} and β_{12} correspond to E_{12}; α_{45} and β_{45} correspond to E_{45}.

[†] Loc. cit., page 170.

[‡] Aside from phraseology, notation, etc.

[§] Loc. cit., pages 170 and 171.

[||] In close parallelism with Osgood I write $r_{in}e_{in}$, $r_{in}'e_{in}'$ instead of α_{in}, β_{in} and avoid the division of β_{in} by α_{in}. There is an evident advantage in this in cases where α_{in} is sometimes 0.

$(i \leqq n)$, $|r_{in}' - r_{in}| \leqq c$. *(b)* $Lim_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$ *exists.* *(c)* E_0 *is a subset of* E *of measure* 0. *(d) If* P *is a point of* E *not belonging to* E_0 *then*

$$\underset{n=\infty}{Lim}\,(r_{i_{Pn}n} - r_{i_{Pn}n}) = 0.$$

CONCLUSION: $Lim_{n=\infty} \sum_{i=1}^{n} r_{in}'e_{in}$ *exists and equals* $Lim_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$.

PROOF. If $\epsilon > 0$ and $\delta > 0$, there exists $n_{\epsilon\delta}$ such that if $n > n_{\epsilon\delta}$ then $S_n \leqq \epsilon$, where S_n denotes the sum of the interior measures of those of the point-sets $E_{1n}, E_{2n}, \cdots, E_{nn}$, for which the corresponding* $|r_{in}' - r_{in}| > \delta$. For otherwise there would be some value ϵ and some infinite sequence of values of n such that, for each value of n belonging to this sequence, $S_n > \epsilon$. Therefore, according to a theorem of W. H. Young's,† generalized so as to apply to space of any finite number of dimensions, there would exist a point set K of interior measure greater than or equal to ϵ, each point of K being contained, for an infinite sequence of different values of n, in some E_{in} for which the corresponding $|r_{in}' - r_{in}| > \delta$. But this would be contrary to Hypothesis (*d*), according to which K must be of measure 0.

It is true, then, that if $\epsilon > 0$ and $\delta > 0$ there exists $n_{\epsilon\delta}$ such that if $n > n_{\epsilon\delta}$ then $S_n \leqq \epsilon$. Hence if $n > n_{\epsilon\delta}$ then

$$\left|\sum_{i=1}^{n}(r_{in} - r_{in}')e_{in}\right| \leqq \epsilon c + \delta e',$$

where e' is the exterior measure of E. Of course by proper choice of ϵ and δ the right hand member of this inequality may be made less than any preassigned positive number. Hence, in view of Hypothesis (*b*), it follows that $\mathrm{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}'e_{in}$ exists and is equal to $\mathrm{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$.

Osgood's Theorem‡ is as follows:

Osgood's Theorem. *Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{A}$$

be a sum of infinitesimals and let α_i *differ uniformly by an infinitesimal of higher order than* Δx_i *from the summand* $f(x_i)\Delta x_i$ *of the definite integral*

$$\int_a^b f(x)dx \tag{B}$$

of the function $f(x)$, *this function being continuous throughout the interval* $a \leqq x \leqq b$. *Then the sum (A) approaches a limit when* $n = \infty$, *and the value of this limit is the definite integral (B):*

$$\underset{n=\infty}{\mathrm{Lim}} \sum_{i=1}^{n} \alpha_i = \int_a^b f(x)dx.$$

---

* r_{i_1n} is said to correspond to E_{i_2n} if $i_1 = i_2$.

† W. H. Young, Lond. Math. Soc. Proc., Ser. 2, vol. II, p. 25.

‡ Loc. cit., page 173.

With regard to ease of application, Theorem 2 has an advantage over Osgood's Theorem in that it is not necessary in applying Theorem 2 to prove that $\text{Lim}_{n=\infty}(r_{in}' - r_{in}) = 0$ *uniformly*.[*]

An Application.—Let us consider a problem to which Osgood applies[†] his Theorem, the problem of finding the attraction of a material body on a particle lying without it. Following Osgood, let the coördinates of the particle P be (a, b, c), let its mass be m, let V be the volume of the material body E and let X be the component along OX of the attraction which the body E exerts on the particle P. Divide the body E into n small pieces E_{1n}, $E_{2n}, \cdots, E_{nn}$ of volumes $\Delta V_{1n}, \Delta V_{2n}, \cdots, \Delta V_{nn}$, respectively, where as $n \doteq \infty$ the maximum diameter of E_{in} approaches 0 as a limit. With[‡] Osgood "Let ρ_{in}'', γ_{in}'', α_{in}'' denote respectively the maximum values of the density ρ in E_{in}, the distance γ from (a, b, c) to a point of E_{in}, and the angle α which a ray drawn from (a, b, c) to a point of E_{in} makes with the positive axis of x; and let ρ_{in}', γ_{in}', α_{in}' denote respectively the minimum values of these functions." Osgood derives the double inequality: [§]

$$k\frac{m\rho_{in}'\Delta V_{in}}{\gamma_{in}''^2}\cos\alpha_{in}'' < \Delta X_{in} < k\frac{m\rho_i''\Delta V_{in}}{\gamma_{in}'^2}\cos\alpha_{in}', \tag{4}$$

where k is a constant of proportionality.

From this point I would proceed as follows:

Let (4)′ designate the double inequality obtained by dividing (4) by ΔV_{in}. If P is any point of E then as $n \doteq \infty$ all points of $E_{i_{P_n}n}$ approach P as a limit. Hence, since ρ, γ and $\cos\alpha$ are continuous functions of (x, y, z) and γ is never 0 in E by hypothesis, therefore

$$\operatorname*{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}'\cos\alpha_{i_{P_n}n}''}{\gamma_{i_{P_n}n}''^2} = \operatorname*{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}''\cos\alpha_{i_{P_n}n}'}{\gamma_{i_{P_n}n}'^2} = \operatorname*{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}\cos\alpha_{i_{P_n}n}}{\gamma_{i_{P_n}n}^2},$$

where $\theta_{i_{P_n}n}$, $\cos\alpha_{i_{P_n}n}$, $\gamma_{i_{P_n}n}$ apply to any arbitrarily chosen point of $E_{i_{P_n}n}$. Hence, from (4)′,

$$\operatorname*{Lim}_{n=\infty}\frac{\Delta X_{i_{P_n}n}}{\Delta V_{i_{P_n}n}} = \operatorname*{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}\cos\alpha_{i_{P_n}n}}{\gamma_{i_{P_n}n}^2}.$$

Clearly the left hand member and the right hand member of (4)′ are bounded. Hence the middle term, $\Delta X_{in}/\Delta V_{in}$, is bounded. It follows, then, from

---

[*] Cf. uniformity condition in Osgood's Hypothesis.

[†] Loc. cit., pp. 166–169 and 174–175.

[‡] Except for double subscripts and other slight changes.

[§] Loc. cit., page 168, double inequality (4).

Theorem 2, that

$$\lim_{n=\infty}\sum_{i=1}^{n}\Delta X_{in}=\lim_{n=\infty}\sum_{i=1}^{n}\frac{km\rho_{in}\cos\alpha_{in}}{\gamma_{in}^{2}}\Delta V_{in},$$

which is equal to

$$\int\!\!\int_{V}\!\!\int k\frac{m\rho\cos\alpha}{\gamma^2}\,dV.$$

Compare this argument with Osgood's argument on page 174, loc. cit.

In this problem if the density ρ should be discontinuous at a set of points E_0 of measure 0 (and continuous elsewhere throughout E) then in the above argument merely stipulate that the point P shall not belong to E_0.

Further Remarks.—The remarks in this section apply to those cases in which as $n \doteq \infty$ the distance between each pair of points that belong to the same E_{in} approaches 0, E is a closed set of points and for each value of n each point of E belongs to some E_{in}.

I wish to discuss three conditions which I will call A, B and C.

A: If P is any point of E then

$$\lim_{n=\infty}(r_{i_{P_n}n}'-r_{i_{P_n}n})=0$$

B: $\lim\limits_{n=\infty}(r_{i_{P_n}n}'-r_{i_{P_n}n})=0$ uniformly as to i.

C: If P is any point of E then $\lim\limits_{n=\infty}(r_{in}'-r_{in})=0$ for all such modes of variation of i with n that, as $n \doteq \infty$, some point of E_{in} approaches[*] P as its limit.

Let A' and C' denote the conditions obtained by substituting $E - E_0$ for E in conditions A and C respectively, where E_0 is a sub-set of E of measure 0.

As has been stated, A is less exacting than B, which corresponds to Osgood's uniformity condition. It may be easily seen however that B is equivalent to C. It is of interest, then, to ascertain whether A' (which I use in Theorem 2) is as exacting as C' which is a condition obtained by weakening (an equivalent of) the uniformity condition B by exempting a set of points (P) of measure 0 from its jurisdiction. The following independence example shows that A' is less exacting than even C'.

---

[*] If the word "is" is substituted here for the word "approaches" then C becomes identical with A. In, for instance, the above problem in mechanics where the density is a continuous function of (x, y, z), A and C could be verified with equal or almost equal facility. But C is equivalent to B which corresponds to Osgood's uniformity condition. These facts are to be considered if one compares Osgood's theorem and mine with regard to ease of application in those cases where no discontinuities enter.

Independence Example.—Assume that the rational points of the interval (0, 1) have, by some scheme, been brought into a one-to-one correspondence with the set of all positive integers. Let t_k denote that rational number which, according to this scheme, corresponds to the positive integer k. Divide the interval (0, 1) into n equal sub-intervals E_{in} by the points x_{in} where $i = 1 \cdots n$. Let $r_{in}' = 1$, only when $n = 2^k$ where k is such a positive integer that $x_{(i-1)n} \leqq t_k \leqq x_{in}$. In all other cases let $r_{in}' = 0$. Ler $r_{in} = 0$ for all values of n and i. Then it may be shown that if P is any point in (0, 1) then in every neighborhood of P there is an interval $E_{i_1n_1}$, such that $r_{i_1n_1}' - r_{i_1n_1} = 0$ and, at the same time, another interval $E_{i_2n_2}'$ such that $r_{i_2n_2}' - r_{i_2n_2} = 1$. Thus condition C is satisfied with respect to *no point whatsoever* of the interval (0, 1). On the other hand condition A' is satisfied with respect to *every point* of that interval.

UNIVERSITY OF PENNSYLVANIA.